TRANSVERSALITY CONDITIOS IN OPTIMUM GROWTH MODELS WITH OR WITHOUT DISCOUNTING: A UNIFIED VIEW

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ABSTRACT

In the literature on price characterization of optimal paths in stationary models of optimal growth, distinct "transversality conditions" have been presented, depending on whether or not utilities are discounted. In the discounted case, this condition takes the form that asymptotically the present-value prices converge to zero. In the undiscounted case, however, it is of the form that the present value prices are bounded above along the path. It is shown here that under assumptions that are fairly standard in such problems, this difference is superfluous and the same transversality condition characterizes optimal paths in both the discounted and undiscounted cases.

SÍNTESIS

En la literatura sobre la caracterización de precios de sendas óptimas en modelos estacionarios de crecimiento óptimo, se manifiestan claras "condiciones de transversalidad", dependiendo de sí se han descontado o no las utilidades. En el caso de las utilidades descontadas, esta condición adopta la forma en que asintóticamente los precios de valor presente convergen a cero. En el caso de las utilidades no descontadas, empero, dicha condición asume la forma en que los precios de valor presente están limitados por sobre la senda. Aquí se demuestra que bajo los supuestos que son habituales para tales problemas, esta diferencia resulta superflua y la misma condición de transversalidad caracteriza las sendas óptimas tanto para los casos de utilidades descontadas como para las no decontadas.

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1. INTRODUCTION

In the literature[1] on price chracterization of optimal paths in stationary models of optimal growth, distinct "transversality conditions" have typically been presented, depending on whether or not the future utilities are discounted. In the discounted case, this condition typically takes the form that asymptotically the present-value prices converge to zero. In the undiscounted case, however, it is of the form that the present value prices are bounded above along the path[2].

This note points out that under assumptions that are fairly standard in such problems, this difference is superfluous and the same transversality condition characterizes optimal paths in both the discounted and undiscounted cases. This result is accomplished by showing that in the discounted case, the apparently weaker form of the limit condition (namely, the present-value prices are bounded) actually implies the stronger form (the present-value prices converge to zero).

2. THE FRAMEWORK

Consider a reduce-form model of optimal growth of the Ramsey type, described by a triplet ($\Im$, u, δ), where $\Im$ is a <u>transition possibility set</u> in $\Re_+^n$ x $\Re_+^n$, with typical element (x, x'), which describes the terminal stocks(x') that can

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[1] See, for instance, Gale (1967), McKenzie (1986), Weitzman (1973) for the reduced form model, and Peleg (1974), and Peleg and Ryder (1972) for a framework in which consumption is treated explicitly.

[2] This, together with the existence of a price supported stationary program satisfying a "strict value-loss property", is used to establish the optimality of a competitive path in the undiscounted case (see, for instance, Theorem 9 of Gale (1967), and Theorem 10.2 of Peleg (1974)).

be technologically attained (in one period) from the initial stocks(x); u is the utility function from $\mathfrak{I}$ to $\Re$; and δ satisfying $o < \delta \leq 1$ denotes the discount factor.[3]

A path from x is a sequence $\{x_t\}$ satisfying $(x_t, x_{t+1}) \in \mathfrak{I}$ for $t \geq 0$, and $x_0 \leq x$. A path $\{x_t\}$ from x is called competitive if there is a price sequence $\{p_t\}$, with p_t in $\Re_+{}^n$ for $t \geq 0$, such that the following "support property" is satisfied for each $t \geq 0$.

$$\delta^t u(x_t, x_{t+1}) + p_{t+1}x_{t+1} - p_{t+t} \geq \delta^t u(x, x') + p_{t+1}x' - p_t x \quad \textit{for all } (x, x') \in \mathfrak{I} \tag{1}$$

A path $\{x_t\}$ from x is called optimal if for every path $\{x_t'\}$ from x, we have

$$\lim_{T \to \infty} \inf \sum_{0}^{T} \delta^t [u(x_t, x_{t+1} - u(x_t', x_{t+1}')] \geq 0 \tag{2}$$

In the undiscounted case ($\delta = 1$), this is the "catching-up criterion" of Gale (1967); in the discounted case ($\delta < 1$), where typically the discounted utility sums are well-defined, this is equivalent to

$$\sum_{0}^{\infty} \delta^t u(x_t, x_{t+1}) \geq \sum_{0}^{\infty} \delta^t u(x_t', x_{t+1}') \tag{3}$$

The following assumptions on the transition possibility set and the utility function will be used:

(A.1) For each $\alpha > 0$, there is $\beta > 0$, such that $(x, x') \in \mathfrak{I}$ and $|x| \leq \alpha$ imply $|x'| \leq \beta$, and $u(x, x') \leq \beta$.

(A.2) There is a number γ such that $(x, x') \in \mathfrak{I}$ and $|x| \geq \gamma$ imply $|x'| \leq |x|$.

(A.3) There is (x, y) in $\mathfrak{I}$ satisfying $y \gg x \geq 0$ ($\mathfrak{I}$ is productive).

[3] Here, $\Re_+{}^n$ denotes the non-negative orthant of $\Re^n$. For x in $\Re^n$, $x \geq 0$ means $x_i \geq 0$ means $x_i \geq$ for each i $= 1, \ldots, n$; $x > 0$ means $x \geq 0$ and $x \neq 0$; $x \gg 0$ means $x_i > 0$ for each i. The norm of x is defined by $|x| = \left[\sum_{1}^{n} (x_i)^2\right]^{1/2}$.

(A.4) (0,0) is in $\mathfrak{I}$ (possibility of inaction).

3. AN EXAMPLE

The general model described above is very convenient because it can be applied to a variety of contexts in which a problem of dynamic optimization arises. To fix ideas, we provide below a typical example which will ilustrate the concepts defined, the assumptions used, and the theorem proved, in a specific context. This is a version of the one-sector model of Neoclassical growth theory (as discussed, for instance, in Lucas (1988) sketched in its simplest form. In keeping with the present context, we adopt a discrete-time formulation of the problem as in Dorfman, Samuelson and Solow (1958) or Koopmans (1957).

There is one produced good which can be used both as a consumption good and as a capital good in production. Let k_t denote the stock of capital input used in producing the output y_t in period t, and c_t the consumption in period t. Since output is distributed over consumption (c_t), net investment (k_{t+1} - k_t), and depreciation (η k_t , assuming a constant depreciation rate $0 \leq \eta < 1$),

$$y_t = c_t + k_{t+1} - k_t + \eta\, k_t \qquad \text{for each } t \geq 0 \qquad (4)$$

No distinction is made between population and labor, which along with capital is an input in production. Taking the simplest case, let the quantity of labor available be exogenously given at a level which is fixed over time. So, even though strictly speaking the production function has two arguments, labor and capital, we may suppress the former and simply write it as f(k), where the function f maps from $\Re_+$ to $\Re_+$. It is understood that the marginal product of capital is positive, decreases to 0 as k becomes very large, and increases to infinity as k becomes very small. A typical example of a production function with these features is the well-know Cobb-Douglas function, $k^a L^b$, where $a > 0$, $b > 0$, and $a+b = 1$. Here, taking L to be fixed and equal to 1(by appropriate choice of units in which labor is measured), $f(k) = k^a$.

A path from k is a non-negative sequence of capital, output and consumption $\{k_t, y_t, c_t\}$ satisfying (4) and

$$y_t = f(k_t) \quad \text{for } t \geq 0 \qquad (5)$$

where k is the initial (historically given) capital stock. It will be convenient to write, combining (4) and (5), that a feasible path is a non-negative sequence $\{k_t, c_t\}$ satisfying

$$f(k_t) + (1-\eta)\, k_t = c_t + k_{t+1} \tag{6}$$

Let $w(c)$ denote the welfare function, so that $w(c_t)$ is the welfare from consumption in period t. It is understood that w exhibits positive and diminishing marginal utility. A standard example of a welfare function having these features is the iso-elastic function: $w(c) = c^{1-\sigma}$, with $0 < \sigma < 1$.

Preferences over feasible paths are given by comparing the (discounted) sum of welfares along paths, using a discount factor, $0 < \delta \leq 1$.

In the case where $\delta < 1$, the discounted welfare sums along paths are finite, and a straightforward comparison can be made to obtain the suitable notion of optimality: a path $\{k_t, c_t\}$ is optimal if for every path $\{k_t', c_t'\}$ from k_0, we have

$$\sum_{0}^{\infty} \delta^t w(c_t) \geq \sum_{0}^{\infty} \delta^t w(c_t') \tag{7}$$

If, however, future generations' welfares are given equal weights in the objective function ($\delta = 1$), the welfare sums typically will not be finite. The usual method of comparison is some form of the overtaking criterion, pioneered by von Weizsacker (1965), which involves comparing welfare sums over arbitrary finite horizons. A path is optimal, loosely speaking, if there in no other path which provides a significantly larger welfare sum for every large (but finite) horizon. Formally, a path $\{k_t, c_t\}$ is optimal if for every path $\{k_t', c_t'\}$ from k_0, we have

$$\lim_{T\to\infty} \inf \left[\sum_{0}^{T} w(c_t) - \sum_{0}^{T} w(c_t')\right] \geq 0 \tag{8}$$

So, we may conveniently write, regardless of which case is being dealt with, that a path $\{k_t, c_t\}$ is optimal if for every path $\{k_t', c_t'\}$ from k_0, we have

$$\lim_{T\to\infty} \inf \left[\sum_{0}^{T} \delta^t w(c_t) - \sum_{0}^{T} \delta^t w(c_t')\right] \geq 0 \tag{9}$$

It is easy to see that the above example is a special case of the general model described in Section 2. In the example, the number of goods, n, equals 1. The capital stock k can be identified with the stock x in the general model. Denote by $\mathfrak{F}$ the set $\{(x, x') : x \geq 0, \text{ and } 0 \leq x' \leq f(x) + (1-\eta)x\}$. For any path $\{k_t, c_t\}$, the sequence $\{x_t\} = \{k_t\}$ satisfies $(x_t, x_{t+1}) \in \mathfrak{F}$ for each $t \geq 0$, by using (6). Define the utility function over $\mathfrak{F}$ by $u(x, x') = w(f(x) + (1-\eta)x - x')$. Then, it is clear that $w(c_t)$ along a path $\{k_t , c_t\}$ is the same as $u(x_t , x_{t+1})$ along the corresponding path $\{x_t\}$ in the general model, and so the notion of optimality as expressed in (9) is the same as that described in (2).

We now comment briefly on the assumptions (a.1) - (A.4) in the context of our example. Assumption (A.1) says that from limited inputs one can get only limited output and limited utility. The value of β (the bound on output and utility) is, of course, not a constant but will typically depend on the value of α (the bound on input). For instance, in our example, (A.1) is satisfied with β equal to the maximum of the two numbers $[f(\alpha) + (1 - \eta) \alpha]$ and $w(f(\alpha) + (1 - \eta) \alpha)$. Assumption (A.2) says that there is enough diminishing returns so that beyond a certain level, stocks cannot be expanded. In our example, the marginal product of capital goes to zero when the capital stock becomes very large, and there is a positive rate of depreciation, so the average product of capital is less than 1 for large capital stocks, and (A.2) is satisfied . Assumption (A.3) makes the model interesting from the economic standpoint in the sense that there are some initial stocks from which expansion is possible, and so positive consumption can be sustained. In the example, the average product of capital is larger than 1 if the capital stock is small enough, and so (A.3) is satisfied. Finally, (A.4) simply states that it is possible to use no input, and thereby produce no output, which is hardly a restriction at all. It is satisfied in our example since f maps from $\Re_+$ to $\Re_+$.

Finally, we briefly interpret the theorem of this paper (see the section below) in the context of the example. It may be verified that in this context the competitive conditions (that is, inequality (1) above) say that the price p_{t+1} of output at the end of each period t, is the discounted marginal utility of consumption, $\delta^t\ w'(c_t)$. Moreover, at these prices, the present value profit in each period (that is, the value of output produced together with that of stocks left over after depreciation, net of costs of inputs employed at the beginning of the period, $p_{t+1} [f(k_t) + (1 - \eta) k_t] - p_t k_t$) is maximized along a competitive path. Existing results in the literature say that such a competitive path which satisfies a transversality condition is optimal. The transversality condition cited is different, however, depending on whether it is the undiscounted case or the discounted case under consideration. In the former case, the condition usually cited reads "the present value of capital, $p_t k_t$, is bounded above", while in the latter it reads "the present value of capital, $p_t k_t$, converges to zero"(see the Proposition in the section below). The theorem of the present paper asserts that the transversality condition in the undiscounted case (namely, "the present value of capital, $p_t k_t$, is bounded above") works in the discounted case as well. Consequently, a single characterization of optimality applies, in principle, to both the discounted and the undiscounted cases.

4. A UNIFYING TRANSVERSALITY CONDITION

We turn, now, to the result of the paper mentioned in the introductory section. Recall from that discussion that what we want to establish is that, in the discounted case, if a competitive path has bounded present value prices then it is optimal[4]. Thus, for the rest of this section, it will be understood that the discount factor satisfies $0 < \delta < 1$.

We note a couple of preliminary results, before coming to the statement and proof of our main theorem. These results are well-known and are, therefore, stated without proofs. First, it follows from assumptions (A.1) and (A.2) that the stock levels and the utility levels obtained along any path starting from a given initial stock are uniformly bounded above by a number which depends only on the given initial stock.

<u>Lemma:</u> Under (A.1) and (A.2), given any initial stock x, there is a number B (depending only on x) such that for any path $\{x_t\}$ from x, $|x_t| \leq B$, and $u(x_t, x_{t+1}) \leq B$ for each $t \geq 0$.

Second, we state formally the usual form of the sufficiency side of the price characterization of optimal paths in the discounted case.

<u>Proposition</u>: Under (A.1) and (A.2), if $\{x_t\}$ is a competitive path from x, with associated supporting prices $\{p_t\}$, and $\lim_{t\to\infty} p_t x_t = 0$, then $\{x_t\}$ is optimal from x.

<u>Remarks</u>: (i) We can replace the condition $\lim_{t\to\infty} p_t x_t = 0$ in the statement of the Proposition by the condition $\lim\inf_{t\to\infty} p_t x_t = 0$. (ii) A change of origin of the utility function leaves essentials unaffected. So, from now on, we take the utility function to be normalized so that $u(0,0) = 0$.

<u>Theorem</u>: Under (A.1) - (A.4), if $\{x_t\}$ is a competitive path from x, with associated supporting prices $\{p_t\}$, and $\lim\inf_{t\to\infty} p_t x_t < \infty$, then $\{x_t\}$ is optimal from x.

<u>Proof</u>: Define the sequence $\{w_t\}$ by

$$w_t = \delta^t u(x_t, x_{t+1}) + p_{t+1} x_{t+1} - p_t x_t \quad \text{for } t \geq 0 \tag{10}$$

[4] Note that we are <u>not</u> concerned here with the <u>necessity</u> side of the price characterization of optimal programs, which involves showing that optimal paths are competitve and satisfy an appropriate transversality condition. This result is well known in the literature and requires suitable convex structures on the model.

Using (A.4) and Remark (ii), we can apply (1) to $(x,x') = (0,0)$ for each $t \geq 0$ to get

$$w_t \geq 0 \quad \text{for } t \geq 0 \tag{11}$$

We now break up the proof by stating and proving two claims.

<u>Claim 1</u>: If $w_t \to 0$ as $t \to \infty$, then $\{x_t\}$ is optimal.

To establish Claim 1, suppose on the contrary that $w_t \to 0$ as $t \to \infty$, but $\{x_t\}$ is not optimal. By the Proposition and Remark (i), we can find $\mu > 0$ and $T \geq 0$ such that $p_t x_t \geq \mu$ for $t \geq T$. By using the Lemma, x_t is bounded above, and so we cand find $\upsilon > 0$, such that

$$|p_t| \geq \upsilon \quad \text{for } t \geq T \tag{12}$$

Using (A.3), (12) and $p_t \geq 0$, if follows that there is a number $m > 0$, such that $p_t(\hat{y} - \hat{x}) \geq m$ for all $t \geq T$, and so

$$p_t \hat{y} \geq p_t \hat{x} + m \geq m > 0 \quad \text{for } t \geq T \tag{13}$$

Applying (1) to $(\hat{x}, \hat{y})$ for $t \geq 0$, we get $w_t \quad \delta^t \; u(\hat{x}, \hat{y}) + p_{t+1} \hat{y} - p$ for $t \geq 0$, by using (10). Then, using (13), we have for $t \geq T$,

$$p_{t+1} \hat{y} \leq p_t \hat{x} + w_t - \delta^t u(\hat{x}, \hat{y}) \leq p_t \hat{y} - m + w_t - \delta^t u(\hat{x}, \hat{y}) \tag{14}$$

Since $x_t \to 0$ and $\delta^t u(\hat{x}, \hat{y}) \to 0$ *as* $t \to \infty$, and $m > 0$, we can find $S \geq T$, such that

$$w_t - \delta^t u(\hat{x}, \hat{y}) - m \leq -(m/2) \quad \text{for } t \geq S \tag{15}$$

Using (14) and (15), we have $p_{t+1} \hat{y} - p_t \hat{y} \leq -(m/2)$ for all $t \geq S$, and so for every $N > S$, we obtain

$$\sum_{t=S}^{N} [p_{t+1} \hat{y} - p_t \hat{y}] \leq -(m/2)(N - S) \tag{16}$$

Simplifying the sum on the left-hand side of (16), we obtain for every $N > S$.

$$-p_S\,\hat{y} \le \quad (p_N - p_S)\,\hat{y} \quad \le -(m/2)\,(N-S) \tag{17}$$

Since (m/2) (N-S) $\to \infty$ as N $\to \infty$, (17) leads to a contradiction, completing the proof of Claim 1.

Claim 2: $\lim\limits_{t\to\infty} inf\; p_t\, x_t < \infty$ *implies that* $\lim\limits_{t\to\infty} inf\; w_t = 0$.

To establish Claim 2, note that by using (10) we can get for each T $\ge$ 0,

$$\sum_0^T w_t = \sum_0^T \delta^t u(x_t, x_{t+1}) + p_{t+1}\, x_{t+1} - p_0\, x_0 \tag{18}$$

Since we know that $\lim\limits_{t\to\infty} inf\; p_t\, x_t < \infty$, we can use (18), the Lemma and $0 < \delta < 1$ to get

$$\lim_{T\to\infty} inf \sum_0^T w_t \le \lim_{T\to\infty} sup \left[\sum_0^T \delta^t u(x_t, x_{t+1})\right] \lim_{T\to\infty} inf\; p_{T+1}\, x_{T+1} < \infty \tag{19}$$

Using (11), we have $w_t \ge 0$ for $t \ge 0$, so (19) implies that $\sum_0^\infty w_t$ is convergent, and so $w_t \to 0$ as $t \to \infty$, establishing Claim 2. Clearly, Claims 1 and 2 together prove the theorem.

Given the boundedness property of the Lemma, it is clear that if the price sequence $\{p_t\}$ (associated with the competitive path $\{x_t\}$ is bounded, then the asymptotic condition of the Theorem is satisfied. this observation yields the following useful corollary.

Corollary: Under (A.1) - (A.4), if $\{x_t\}$ is a competitive path from x, with associated supporting prices $\{p_t\}$, and $\lim\limits_{t\to\infty} inf\; \|p_t\| < \infty$ then $\{x_t\}$ is optimal from x.

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